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Inverse Scattering: Ionospheric Structure Determination

**Jeffrey M. Cohen
Physics Department
University of Pennsylvania
Philadelphia, Pa. 19104**

**August 1980
Final Technical Report
for period 1 July 1979 to 30 June 1980**

**Air Force Office of Scientific Research
Directorate of Mathematical and Information Sciences
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1 July 1979 to 30 June 1980

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1. Accomplishments under Contract [REDACTED] for the Period
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1 July 1979 to 30 June 1980

1.1 Papers and Lectures

a. Comparison of the Approximate and Exact Full Wave
Theory for the Sounding of a Stratified Ionosphere

i. University of Pennsylvania Report

ii. Report Enclosed with Final Report

b. Colloquium, University of Pennsylvania, February, 1980

Colloquium, New Jersey Institute of Technology,
December, 1979

1.2 Reserach Objectives Which Have Been Accomplished

A copy of the above-mentioned report is included with this final report. A paper entitled "Curved Space Scattering" has already been sent to AFOSR. The abstracts of the paper and report contain statements of the research accomplished. In addition we have looked into the following problems:

- a. We have employed the Kanal-Moses variational principle to treat the synthetic data discussed in the report listed in 1.1. It was found that if the initial trial function is within 10% of the actual result, then the K-M variational principle gives the result to better than 1%.
- b. We have generalized the results of Kay for n-poles in such a way that practical applications are possible, e.g. to the ionosphere. To test the method we are in the process of studying the 4- and 10-pole cases before treating the 100-pole case.

1.3 Interaction with Other Investigators

- a. We discussed scattering at a Physics Colloquium, University of Pennsylvania, February, 1980.
- b. We have been discussing problems of mutual interest on a continuing basis with Prof. H. Moses of the University of Lowell and Dr. A. Skalafuris of the Naval Research Laboratory. The problem of obtaining information beyond a potential peak was discussed with Prof. Percy Deift of the Courant Institute and with Dr. Robert Greene of Science Applications Inc. We have had extensive discussions concerning inverse scattering with Prof. C. V. Vishveshwara of the Raman Institute in Bangalore, India. A collaboration between India and the United States is now under way. Applications to the ionosphere have been discussed with Drs. A. Jordan and S. Ahn of the Naval Research Laboratory. Applications to oceanography have been discussed with Dr. E. Toton of the Naval Oceanographic Laboratory and Dr. R. Adams of the Applied Physics Laboratory.

A COMPARISON OF THE APPROXIMATE AND EXACT FULL WAVE THEORY
FOR THE SOUNDING OF A STRATIFIED IONOSPHERE

Jeffrey Cohen and Michael Kearney
Department of Physics, University of Pennsylvania
Philadelphia, Pennsylvania 19104

1. INTRODUCTION AND SUMMARY

The purpose of the present communication is to compare methods of obtaining the electron density profile $N(z)$, which is assumed to be a function of altitude z , using the usual approximate W.K.B. method and using the full-wave method. The data needed to compute $N(z)$ using the two methods is identical. This data is the time as a function of frequency which it takes a train of horizontally polarized electromagnetic waves, transmitted vertically, to be reflected from the ionosphere back to the transmitter.

Since the data used is identical for the approximate and full-wave theories, there is, in principle at least, no need to modify equipment. The difference in treatments is essentially computational and the same experiment can be used to obtain $N(z)$ using the approximate and exact theories.

The approximate treatment has been in use for over forty years and many of the purely computational difficulties have long since been overcome. Those who use ionosondes and calculate profiles have become so accustomed to the use of the approximate method that they often forget the method really is approximate

The exact treatment suffers from the defect that it is new and therefore the purely computational methods are only now being set up. Indeed one of the objects of the present communication is to stress the need for understanding the computational difficulties and thus make it possible to apply the full wave theory to the data. But the principal point of this introduction, and of this communication is to provide a larger view of the problem.

2. THE EXACT DIFFERENTIAL EQUATION FOR THE ELECTROMAGNETIC FIELD.

Let $E(z,t)$ be a component of the electric field perpendicular to the z -axis or vertical axis along which the electron density is stratified. Then with

$$E_k(z,t) = e^{i\omega t} E_k(z) ,$$

$$(\omega = ck) \tag{1}$$

the equation for $E_k(z,t)$ is (ref. 1, page 129)

$$\frac{d^2}{dz^2} E_k(z) + k^2 n^2 E_k(z) = 0 . \tag{2}$$

In (1) ω is the angular frequency, $k = \frac{\omega}{c}$ is the wave number in free space and n is the index of refraction given by

$$n^2 = 1 - \frac{N(z) 4\pi e^2}{mc^2 k^2} \tag{3}$$

where m , e , c are the mass of the electron, the charge of the electron and velocity of light in Gaussian cgs units.

In our discussion we have ignored electron collisions and the effect of the earth's magnetic field. This physical (as opposed to mathematical) approximation is often assumed in ionospheric sounding methods. At the magnetic equator, the earth's magnetic field is parallel to the earth's surface and hence the polarization of the ionosonde can be chosen so that the magnetic field does not affect the scattered wave. In other situations, the effect of the magnetic field may also be eliminated (see Ref. 3). The effect of collisions is negligible at higher altitudes ($z > 80$ km). One of the ultimate objectives of the present train of research is to determine whether these physical approximations lead to errors greater than those made by the mathematical approximations. In any case, we shall assume as

is customary that the use of Eq. (3) for the index of refraction is a good approximation for use in ionospheric sounding in certain geographical regions.

Equation (2) can also be written

$$\frac{d^2}{dz^2} E_k(z) + (k^2 - V(z)) E_k(z) = 0 \quad (4)$$

where $V(z)$ is given by

$$V(z) = K N(z), \text{ with } K = \frac{4\pi e^2}{mc^2}. \quad (4a)$$

Equation (4) is the one-dimensional Schroedinger equation which has been exhaustively studied. There has been a renewal of interest arising in recent years because of its connection with soliton theory (4). The potential $V(z)$ is essentially the number density $N(z)$.

In the direct problem of reflecting electromagnetic waves from the ionosphere we assume $V(z)$ (or $N(z)$) is very small for $z < z_0$ and $z > z_1$.

We look for solutions of (4) which behave like

$$\begin{aligned} E_k(z) &= e^{ikz} + b(k) e^{-ikz} \text{ for } z < z_0 \\ &= t(k) e^{ikz} \text{ for } z > z_1 \end{aligned} \quad (5)$$

The quantities $b(k)$ and $t(k)$ are called the reflection and transmission coefficient respectively.

For a wave with wave number k ,
we have

$$\begin{aligned} E(z,t) &= e^{ik(z-ct)} + b(k) e^{-ik(z+ct)}, \quad z < z_0 \\ &= t(k) e^{ik(z-ct)}, \quad z > z_1. \end{aligned} \quad (6)$$

The boundary conditions (b) are interpreted to mean that a plane wave with wave number $k = \frac{2\pi}{\lambda}$ moves initially toward the scattering potential $V(z)$ and then is partially reflected toward the left. A transmitted wave on the other side of the potential which moves to the right is also present.

Actually, one never sends an infinite plane wave toward the potential (or ionosphere). Instead one sends a pulse containing several wave lengths. This pulse can be represented by

$$E(x,t) = \int_{-\infty}^{+\infty} A(k) e^{-i\omega t} E_k(z) dk \quad (7)$$

and is thus a superposition of the infinite plane waves. The amplitude factor $A(k)$ has its peak value near or at the value of $k = k_0$, where k_0 is the wave number of the plane wave which appears in the pulse. One sends in a pulse

$$E_{\text{incident}}(x,t) = \int_{-\infty}^{+\infty} A(k) e^{ik(x-ct)} dk \quad (8)$$

and gets back a reflected pulse

$$E_{\text{reflected}}(x,t) = \int_{-\infty}^{+\infty} A(k) b(k) e^{-ik(x+ct)} dk. \quad (9)$$

Since $E_{\text{incident}}(x,t)$ and $E_{\text{reflected}}(x,t)$ are of finite extent we can ask for the time for the reflected wave to return to the transmitter. This time will depend on k_0 . It is given by

$$T(k_0) = \frac{1}{c} \left[\frac{\partial \phi(k)}{\partial k} \right] \text{ evaluated at } k = k_0 \quad (10)$$

where $\phi(k)$ is the phase of $b(k)$; i.e.

$$b(k) = |b(k)| e^{i\phi(k)}. \quad (11)$$

The virtual height h_v is given by

$$h_v = \frac{1}{2} cT(k_0). \quad (11a)$$

It should be remembered that the theory thus far is exact.

(In using (10) for the time delay it is convenient to think of the transmitter as being located at $z = 0$.)

3. THE APPROXIMATE INVERSE PROBLEM

The direct problem is, for our purposes, the following:

Given $V(z)$ (or equivalently $N(z)$), find $T(k)$.

The inverse problem is: Given $T(k)$ for all k , find $V(z)$ (or $N(z)$).

The approximate solution is obtained from the W.K.B. approximation. Assume $E_k(z)$ has the form

$$E_k(z) = A e^{iF(z)} \quad (12)$$

Substitute (12) into equation (2) to obtain

$$\left(\frac{dF}{dz}\right)^2 = k^2 n^2 + i \frac{d^2 F}{dz^2} \quad (13)$$

which becomes a Riccati equation if $y = dF/dz$. An iteration procedure can be started by assuming on the right of eq. (13) that $k^2 n^2$ is large compared to $\frac{d^2 F}{dz^2}$, i.e. n/λ is large or n varies slowly compared to the wave length (see Ref. 3 for a more careful analysis of domain of validity). The first iteration gives $F' = \pm kn$ while the second yields

$$F' = \pm kn \left(1 + \frac{iF''}{k^2 n^2}\right)^{\frac{1}{2}} \approx \pm kn \left(1 + \frac{iF''}{2k^2 n^2}\right) = \pm kn + \frac{i}{2n} n'$$

which gives F by a quadrature. Since Eq. (13) is non-linear, the two particular solutions of (13) cannot be superimposed to find a general solution. However, the solution F can be substituted into Eq. (12) to obtain particular solutions

$$E_k(z) = A [n(z)]^{-\frac{1}{2}} \exp \left[\pm i k \int_{z_0}^z n(z') dz' \right] \quad (14)$$

to Eq. (2). If one particular solution to Eq. (2) is known, it can be used to give a first order linear equation which can be solved by quadratures thereby giving the general solution.

Alternatively, two distinct particular solutions can be added to give the general solution of Eq. (2) which is (within this approximation)

$$E_k(z) = A(k) [n(z)]^{-1/2} \exp \left[+ i k \int_{z_0}^z n(z') dz' \right] + B(k) [\underline{n}(z)]^{-1/2} \exp \left[- i k \int_{z_-}^z n(z') dz' \right]. \quad (15)$$

Using the first boundary condition of Eq. (6) and noting $n(z) = 1$ for $z \leq z_0$, one finds

$$\begin{aligned}
E_k(z) &= e^{ikz} + b(k) e^{-ikz} \quad (z < z_0) \\
&= A(k) e^{-ikz_0} e^{ikz} + B(k) e^{ikz_0} e^{-ikz} .
\end{aligned}$$

Hence $A(k) = e^{ikz_0}$

$$h(k) = b(k) e^{ikz_0} \quad (16)$$

Let z_k be the value of z for which $n(z) = 0$, i.e. from Eq. (3) and (4a)

$$V(z_k) = k^2 . \quad (17)$$

Our picture, the usual one, is that $V(z) = 0$ for $z < z_0$ and increases monotonically toward a maximum as z increases above z_0 . As k^2 varies from 0 to the maximum of $V(z_k)$, a z_k is defined by (17).

From (15), $E_k(z_k) \rightarrow \infty$ unless

$$\begin{aligned}
A(k) \exp\left[i k \int_{z_0}^{z_k} n(z') dz'\right] + B(k) \exp\left[-i k \int_{z_0}^{z_k} n(z') dz'\right] \\
= 0.
\end{aligned} \quad (18)$$

Thus

$$\begin{aligned}
B(k) &= -A(k) \exp\left[2ik \int_{z_0}^{z_k} n(z') dz'\right] \\
&= -e^{ikz_0} \exp\left[2ik \int_{z_0}^{z_k} n(z') dz'\right] .
\end{aligned} \quad (19)$$

Or from (16)

$$b(k) = -e^{2ikz_0} \exp\left[2ik \int_{z_0}^{z_k} n(z') dz'\right] . \quad (20)$$

The phase $\phi(k)$ of Eq. (11) is

$$\phi(k) = \pi + 2 k \left[z_0 + \int_{z_0}^{z_k} n(z') dz' \right] . \quad (21)$$

$$T(k) = \frac{2}{c} \left[z_0 + \int_{z_0}^{z_k} n(z') dz' \right] + \frac{2k}{c} \int_{z_0}^{z_k} \frac{\partial}{\partial k} n(z') dz' + \frac{2k}{c} n(z_k) \frac{\partial z_k}{\partial k} . \quad (22)$$

But $n(z_k) = 0$ by definition (see Eqs. (3) and (4)), also

$$\begin{aligned} \frac{\partial n(z)}{\partial k} &= \frac{\partial}{\partial k} \frac{1}{k} \sqrt{k^2 - V(z)} \\ &= -\frac{1}{k^2} \sqrt{k^2 - V(z)} + \frac{1}{\sqrt{k^2 - V(z)}} \\ &= -\frac{1}{k} n(z) + \frac{1}{kn(z)} . \end{aligned} \quad (23)$$

Thus finally

$$T(k) = \frac{2}{c} z_0 + \int_{z_0}^{z_k} \frac{dz'}{n(z')}$$

or

$$T(k) = \frac{2}{c} \left[z_0 + k \int_{z_0}^{z_k} \frac{dz'}{\sqrt{k^2 - V(z')}} \right] . \quad (24)$$

This is a well-known integral equation for $V(z)$. The first term on the right $\frac{2z_0}{c}$ represents the time for the signal to reach z_0 and return to the transmitter at $z = 0$. If k_0 is the lowest value of k for which there is a reflection, then

$$z_0 = \frac{cT(k_0)}{2} . \quad (25)$$

Now that z_0 is determined, one can find $V(z)$ (or $N(z)$) using Abel's method of inversion (Ref. 5). For each value of k one gets $T(k)$ experimentally. One then can find $V(z)$ ($z_0 < z < z_k$) using Abel's method. This is the usual procedure.

Note if $k^2 > V_{\max}(z)$, the method fails. Even if k^2 is near but less than V_{\max} , we are outside the domain of validity because $V(z)$ changes rapidly near its maximum.

4. THE EXACT INVERSE PROBLEM

The exact or full-wave theory version of the inverse problem is developed in Refs. 1 and 2.

Let $b(k)$ be the reflection coefficient as in Eq. (6). The potential can be recovered from a knowledge of $b(k)$ through the use of the Gelfand-Levitan algorithm. To be specific let us define $R(z)$ by

$$R(z) = (2\pi)^{-1} \int_{-\infty}^{+\infty} b(k) e^{-ikz} dk. \quad (26)$$

The reflection coefficient $b(k)$ satisfies the following conditions (see Refs. 1, 2 and 6.)

$$b(-k) = b^*(k), \quad b(k) \text{ analytic in the upper half-plane,}$$

$$b(0) = -1, \quad R(z) \equiv 0 \text{ for } z < 2z_0, \quad b(k) = e^{2ikz_0} g(k). \quad (27)$$

The phase of $g(k)$ approaches a finite limit as $|k| \rightarrow \infty$ on the real axis. The time delay obtained from the phase of $b(k)$, approaches the value $\frac{2z_0}{c}$ as $|k| \rightarrow \infty$, i.e. if $\phi(k)$ is the phase of $b(k)$ as in Eq. (11)

$$\phi(k) \rightarrow 2kz_0 \text{ as } |k| \rightarrow \infty. \quad (28)$$

Let us define the Gelfand-Levitan kernel $K(z, y)$ by

$$K(z, y) = 0 \text{ for either } y > z, \text{ or } z < z_0. \quad (29)$$

For $y < z$ and at the same time $z > z_0$ we require $K(z, y)$ to satisfy the Gelfand-Levitan equation:

$$K(z, y) = -R(z+y) - \eta(z+y-2z_0) \int_{2z_0-y}^x K(z, u) R(u+y) du \quad (30)$$

where $\eta(x)$ is the Heaviside function defined by $\eta(x) = 1$ for $x > 0$, $\eta(x) = 0$ for $x < 0$.

Having found the Gelfand-Levitan kernel $K(z,y)$, $V(z)$ is given by the simple expression

$$V(z) = 2 \frac{d}{dz} K(z,z) . \quad (31)$$

However, the electric field $E_k(z)$ can also be obtained using

$$E_k(z) = e^{ikz} + b(k) e^{-ikz} + \int_{2z_0-z}^z K(z,u) [e^{iku} + b(k) e^{-iku}] du. \quad (32)$$

Thus to find $V(z)$ using the algorithm, we obtain the reflection coefficient $b(k)$ from its phase $\phi(k)$.

This is accomplished using the analytic properties of $b(k)$ and hence $\log b(k)$ and a generalized form of the Hilbert transform. To be specific, $(d/dk)\phi(k)$ is found from the time delay in accordance with Eq. (10). From Eq. (28) z_0 can be found. The phase $\phi(k)$ is obtained by integration with the boundary condition $\phi(0) = \pi$. Let $v(k)$ be defined by

$$v(k) = \phi(k) - 2kz_0 \quad (33)$$

and $w(k)$ by

$$w(k) = \log |b(k)| \quad (34)$$

then

$$w(k) = \frac{P}{\pi} \int_{-\infty}^{+\infty} \frac{kv(k')}{k'(k'-k)} dk'. \quad (35)$$

In Eq. (35), the symbol P means the principal part of the integral.

It should be mentioned that there are variational principles available (Ref. 7) which enable one to obtain $V(z)$ when $b(k)$ is known. These principles have an upper bound built into them.

One also has available a generalization of the Gelfand-Levitan algorithm. If for a given reflection coefficient $b_0(k)$ one knows the electron density $V_0(z)$, one can obtain $V(z) - V_0(z)$ in terms of $b(k) - b_0(k)$ (Ref. 8). One can view this generalization as offering at least two options. One may regard $V(z) - V_0(z)$ as the error in density due to an error $b(k) - b_0(k)$ in the reflection coefficient. Or one may think of $V_0(x)$ as being the density associated with a time delay leading to $b_0(k)$ having been obtained from a model or a previous calculation (even using the WKB method). Then $V(x)$ is obtained as a relatively small change due to the change in the reflection coefficient. The variational principle can also be used to obtain $V(x) - V_0(x)$ from $b(k) - b_0(k)$ together with a bound on this difference. We refrain from details.

5. NUMERICAL SOLUTION OF THE EXACT INVERSE PROBLEM

A straightforward technique for solving the Gelfand-Levitan integral equation numerically is to replace equation (30) with a discrete approximation on a uniform coordinate grid in the (z,y) plane. (Refs. 10 and 11). Assuming nonvanishing z_0 and grid size Δ yields the approximation

$$K_{mn} = -R_{m+n} - \Delta \sum_{i=0}^{m+n} K_{m,i-n} R_i W_{i,m+n} \quad -m < n < n \quad (36)$$

where $K_{mn} = K(m\Delta, n\Delta)$ and $R_m = R(m\Delta)$. The term $W_{i,m+n}$ is a weighting factor which depends on which quadrature rule is chosen to approximate the integral in equation (30). The best choices would be those forms which maximize the precision of the approximation for a given grid size and integrand and which allow stable numerical computation. Note that the weighting factor may depend on the number of points summed over. For example many quadrature rules depend on whether the number of grid points summed over is odd or even.

The discrete fourier transform (Ref. 12) provides a natural framework for calculating the values of R_m , the fourier transform of the reflection coefficient, $b(k)$, at the coordinate grid points. This follows from the fact that fourier integrals can be approximated by discrete sums over a uniform grid and that computationally efficient algorithm, the fast fourier transform (FFT), (Ref. 14) exists for evaluating these sums (Ref. 13).

Once the values of $R_{mi}, i=0, M$ have been calculated (we assume here that $b(k)$ is given or measured experimentally) Eq. (36) is

equivalent to a set of $2m+1$ simultaneous equations for each value of m . Solution of these equations will yield all the values of K_{mn} for $-m \leq n \leq m$. If desired, scattering potential may then be derived from the values of K_{mm} , using eq. (31).

There are a variety of techniques available for solving such sets of simultaneous equations. These techniques may generally be divided into two classes, direct and iterative. Direct methods have the advantage of assured solution for nonsingular matrices but require $\sim N^3$ arithmetic operations for their solution (for matrices of size N). This is a distinct disadvantage for large sets of equations. Iterative techniques (Gauss-Seidel, Jacobi, for example), generally require KN^2 operations (k is the number of iterations) but do not assure convergence in less than N iterations and so may not be superior to direct methods. Accelerated convergence might be accomplished through the use of relaxation methods, a common approach in the numerical solution of partial differential equations. However, the best technique probably depends to a great extent on the problem at hand.

The authors have used both approaches successfully on the model calculations of the next section.

6. NUMERICAL SOLUTION FOR SOME MODEL POTENTIALS

Our approach to testing the numerical methods described in the previous section is to use synthetic data obtained from simple potential models for which exact, analytic solutions are available for the potential, reflection coefficients and eigenfunctions. This has the advantage that the potential $V(x)$ and hence the Gelfand-Levitan kernel $K(x,x)$ can be evaluated easily. This gives a direct, simple technique for evaluating numerical methods for accuracy, stability and speed.

We have chosen to start with model reflection coefficients derived from known potentials rather than from arbitrary reflection coefficients corresponding to a potential with properties not known a priori. The solutions of the Gelfand-Levitan integral equation may then be compared with the known solutions in a straightforward way.

in table I

Below, we list the models we have used for preliminary tests of our numerical techniques. We give both the potential form and reflection coefficient. For each potential we have calculated the virtual height (Eq.(11a) and the fourier transform of the reflection coefficient for parameters which are typical of those observed in the ionosphere: penetration frequencies of the order $\sim 3\text{mhz}$, widths of order $\sim 50\text{ km}$. These results are presented in Figs. 1 through 14. One feature of the fourier transforms should be mentioned. For the more realistic ionospheric models n , the rectangular and the parabolic electron distributions, the transforms exhibit rapid oscillations whose wavelength is roughly K_p^{-1} ,

where K_p is the penetration wave number. This feature will place a severe strain on calculations using the discrete Gelfand-Levitan equation, as the grid size must be made sufficiently small so that the approximation for the integral in (30) is accurate. This means that one will have to solve a large set of simultaneous equations in order to produce the desired results. The solution of such sets of equations is a machine intensive product requiring large amounts of storage. At this time, it is this factor which restricts the major possible numerical solutions of the Gelfand-Levitan equation. The authors are presently investigating the possibility of lifting or at least easing this restriction.

Figures 12 through 15 present the results for $K(x,x)$ versus x for four potential models: a single delta function (12), a potential with a rescaled delta function reflection coefficient (14), a 2-delta-function potential (13), and a potential step (15). The results exhibit two major properties. First the presence of discontinuities in $K(x,x)$ does not impair the effectiveness of the algorithm, as the discontinuities one expects from the analytic solutions for $V(x)$ are present in the numerical results. Second, the precision of the results gets worse as x increases. This is to be expected since any quadrature rule used to approximate the integral in (30) will be less accurate, the larger the range of integration. This effect can be reduced by keeping the grid size small and using those approximations which are best suited to the form of the integrand.

TABLE I - IONOSPHERE MODELS

Model I - Single Delta Function

Potential	$V(x) = V_0 \delta(x)$
Reflection Coefficient	$b(k) = -i((2k/V_0) + i)^{-1}$
Fourier transform of $b(k)$	Figure 7
Virtual height	Figure 1
Gelfand-Levitan Solution	Figure 12

Model II - Rescaled Delta Function

Potential	$V(x) = vV_0 \delta(x) - V_0 \sigma^2 \operatorname{sech}^2(\sigma x + \alpha)$ $\sigma = (1-v^2)^{1/2} \sinh \alpha = ((1-v)/2v)^{1/2}$
Reflection Coefficient	$b(k) = -iv((2k/V_0) + i)^{-1}$
Fourier Transform of $b(k)$	Figure 1 (rescale by v)
Virtual height	Figure 2
Gelfand-Levitan Solution	Figure 14

Model III - Double Delta Function

Potential	$V(x) = V_0 \delta(x) + V_1 \delta(x - x_1)$
Reflection Coefficient	$b(k) = (-(1 + (2ik)/V_0) + (1 - (2ik/V_1))e^{-2ikx_1}) \times$ $(-(1 - (2ik/V_0))(1 - (2ik)/V_1)e^{-2ikx_1} + 1)^{-1}$
Fourier Transform of $b(k)$	Figure 8
Virtual Height	Figure 2
Gelfand-Levitan Solution	Figure 13

Model IV - Potential Step

Potential $V(x) = \eta(x)V_0$

Reflection Coefficient $b(k) = \begin{cases} (1-i(\beta^{-2}-1)^{\frac{1}{2}})/(1+i(\beta^{-2}-1)^{\frac{1}{2}}), \beta < 1 \\ (1-(1-\beta^{-2})^{\frac{1}{2}})/(1+(1-\beta^{-2})^{\frac{1}{2}}), \beta \geq 1 \end{cases}$

$$k_0 = V_0^{\frac{1}{2}}, \beta = k/k_0$$

Fourier Transform
of $b(k)$

Figure 9

Virtual Height

Figure 3

Gelfand-Levitan
Solution

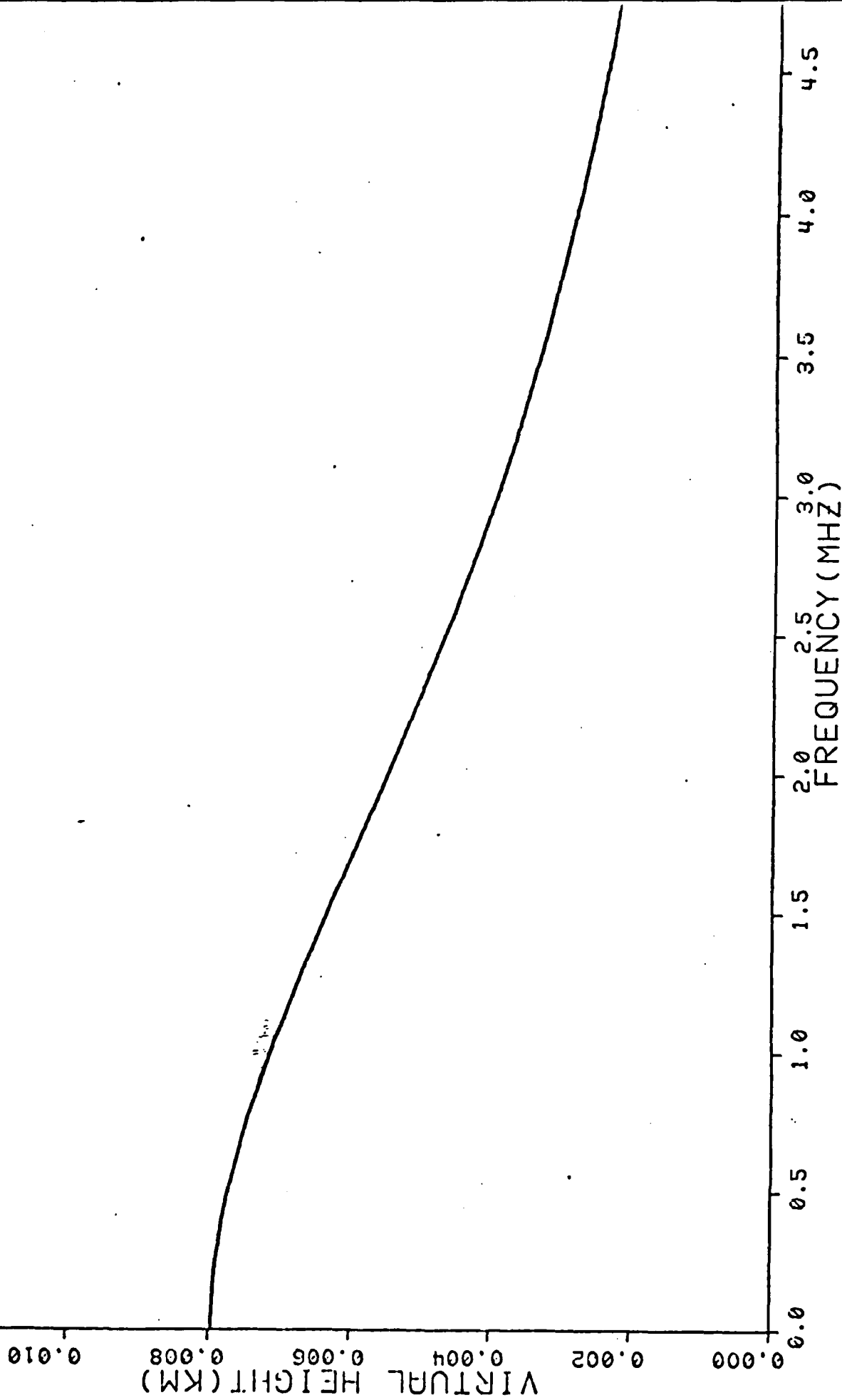
Figure 15

References

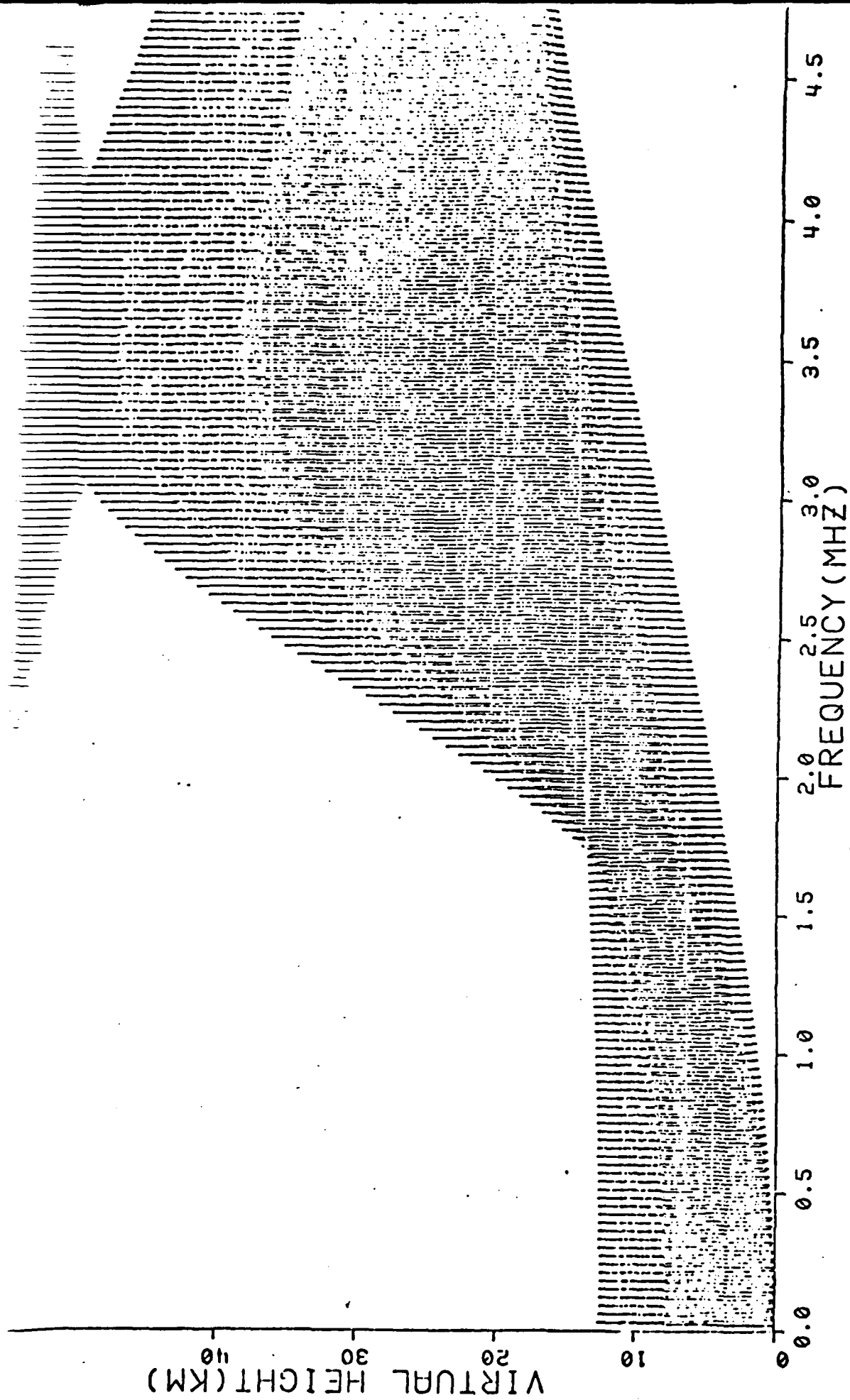
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Acknowledgement

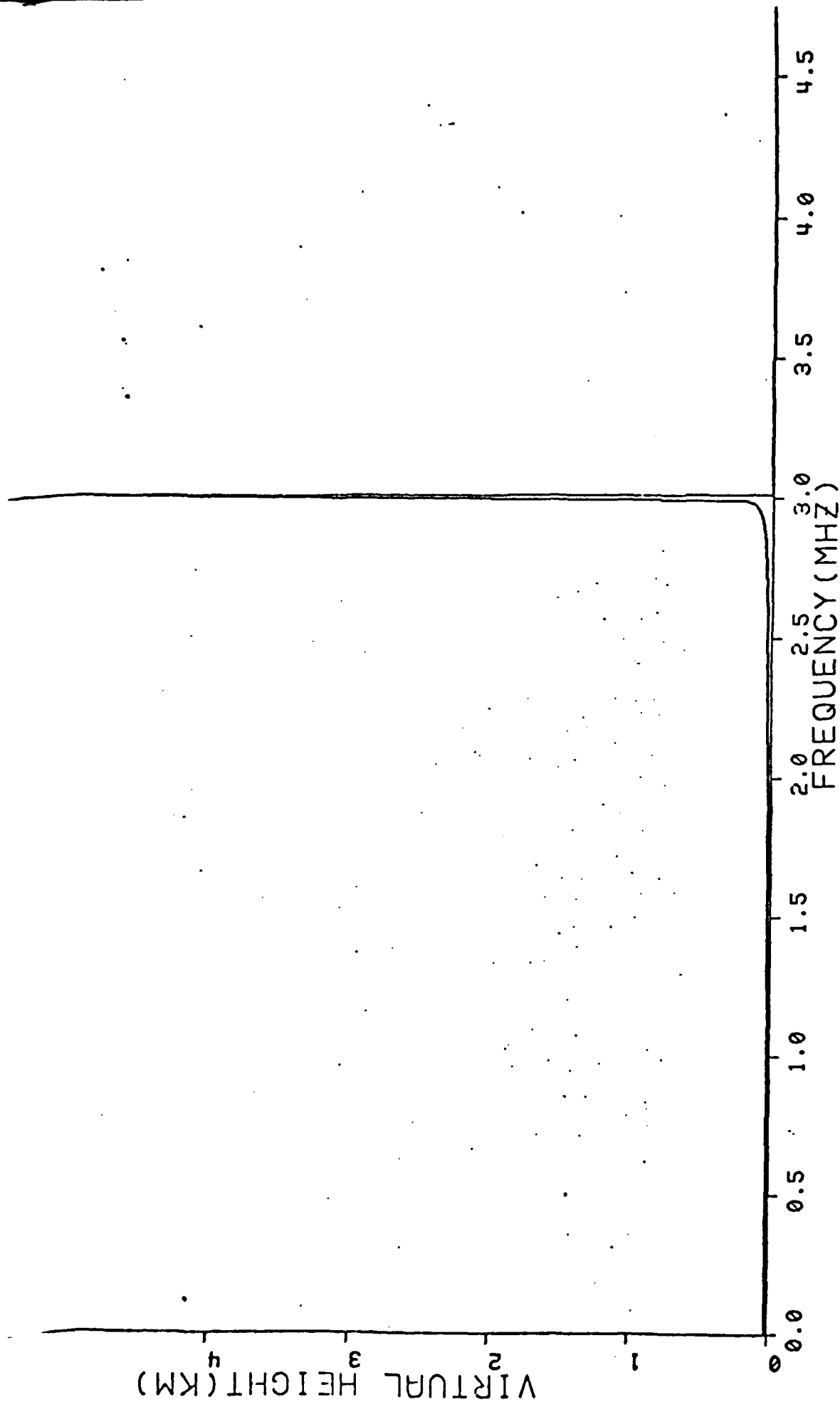
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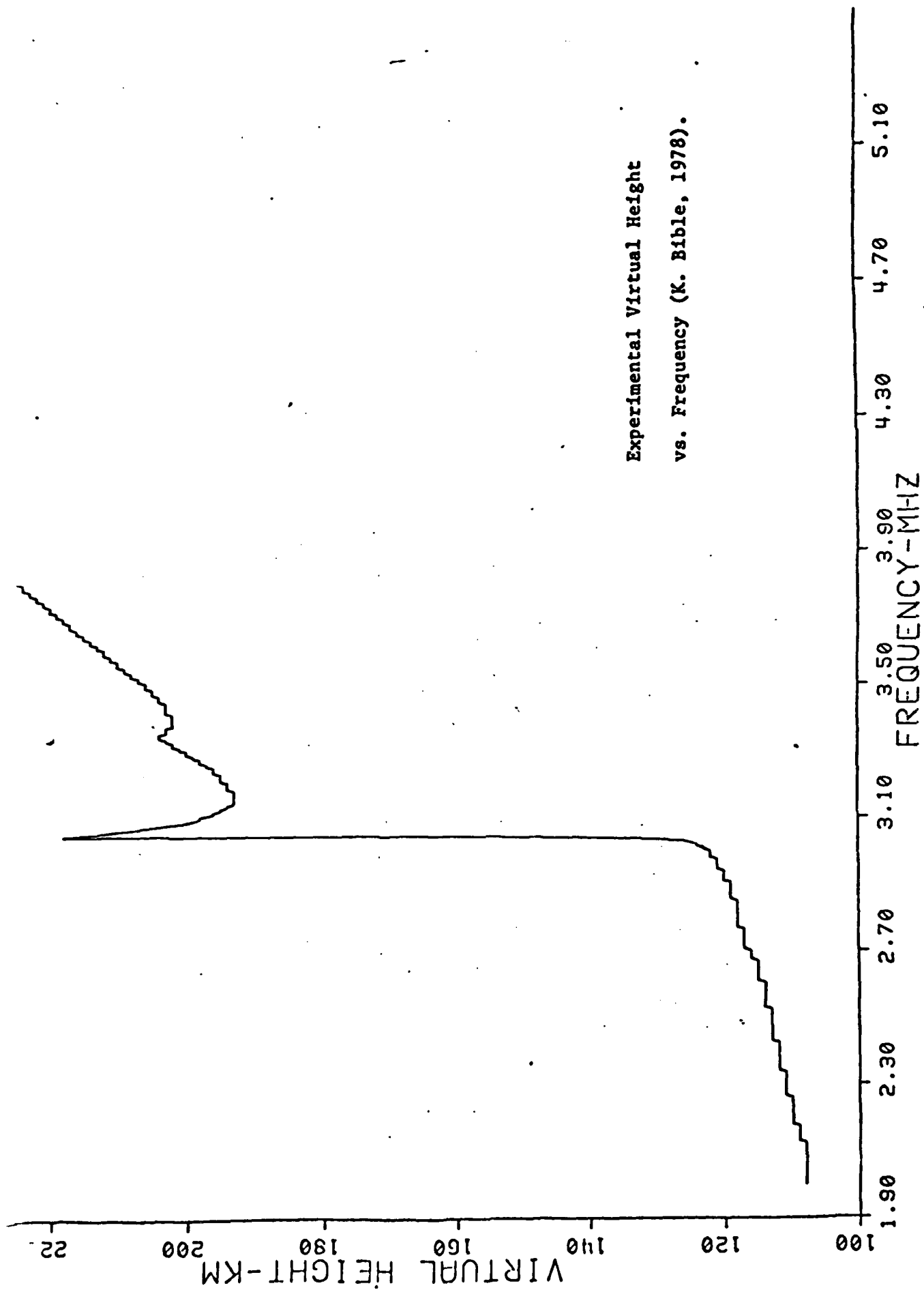
VIRTUAL HEIGHT VS FREQUENCY FOR REFLECTION FROM A DELTA FUNCTION DISTRIBUTION OF ELECTRON DENSITY. penetration frequency - 3MHZ, width--not applicable

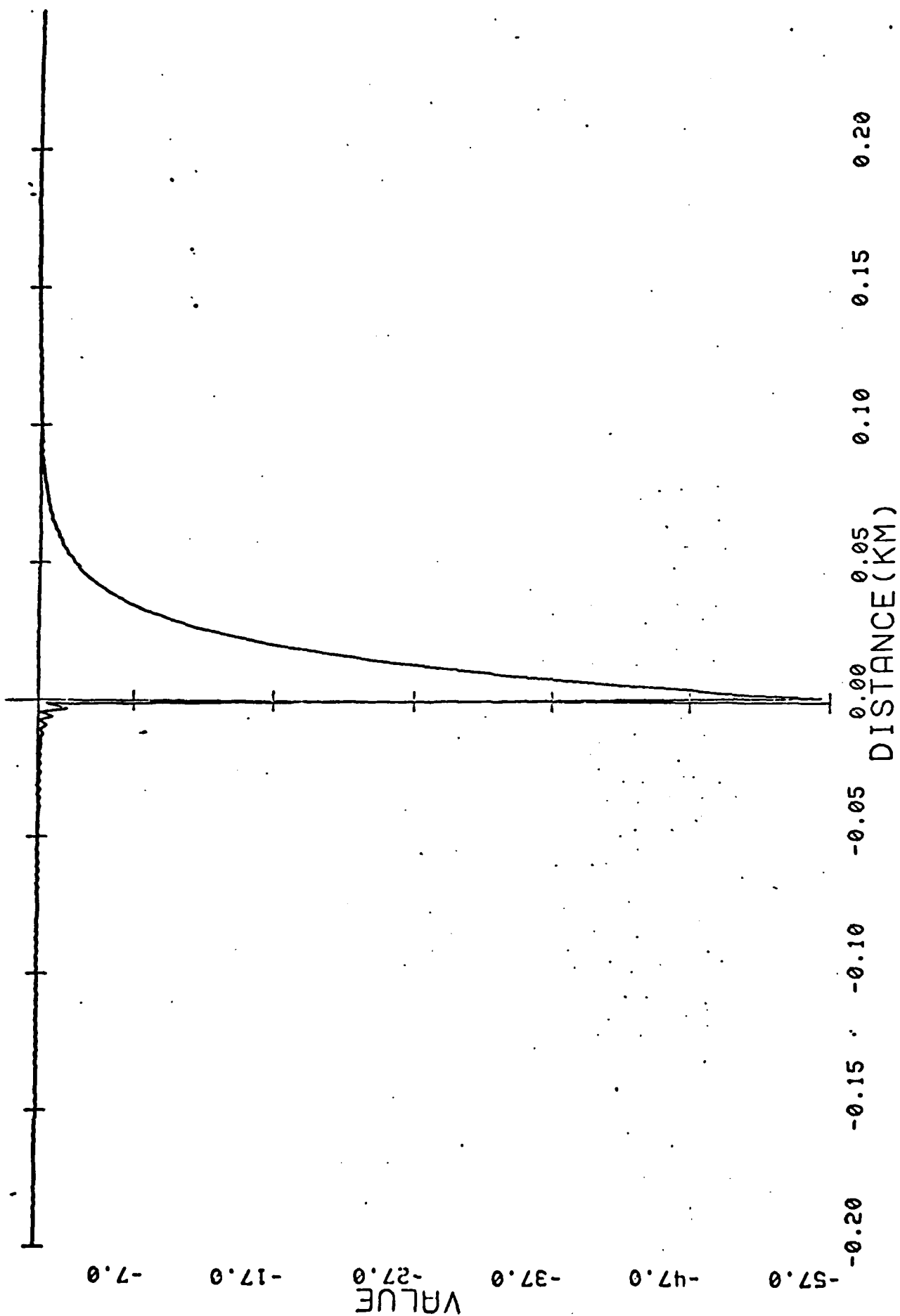


VIRTUAL HEIGHT VS FREQUENCY FOR REFLECTION FROM A 2 DELTA FUNCTION DISTRIBUTION
IN ELECTRON DENSITY penetration frequency = 3MHZ, width = 50KM

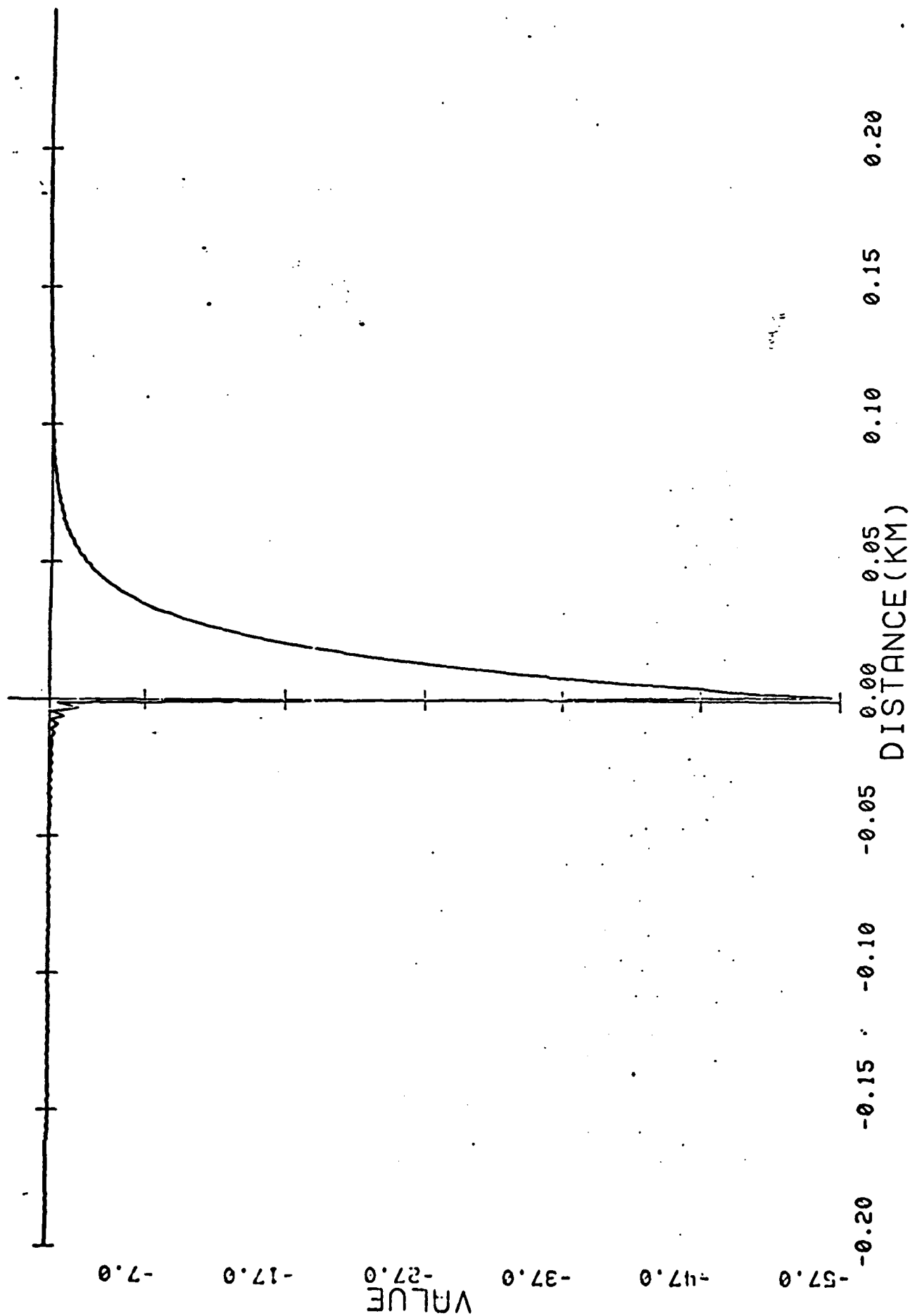


VIRTUAL HEIGHT VS FREQUENCY FOR REFLECTION FROM A STEP IN ELECTRON DENSITY
penetration frequency= 3MHz,width-not applicable for this model

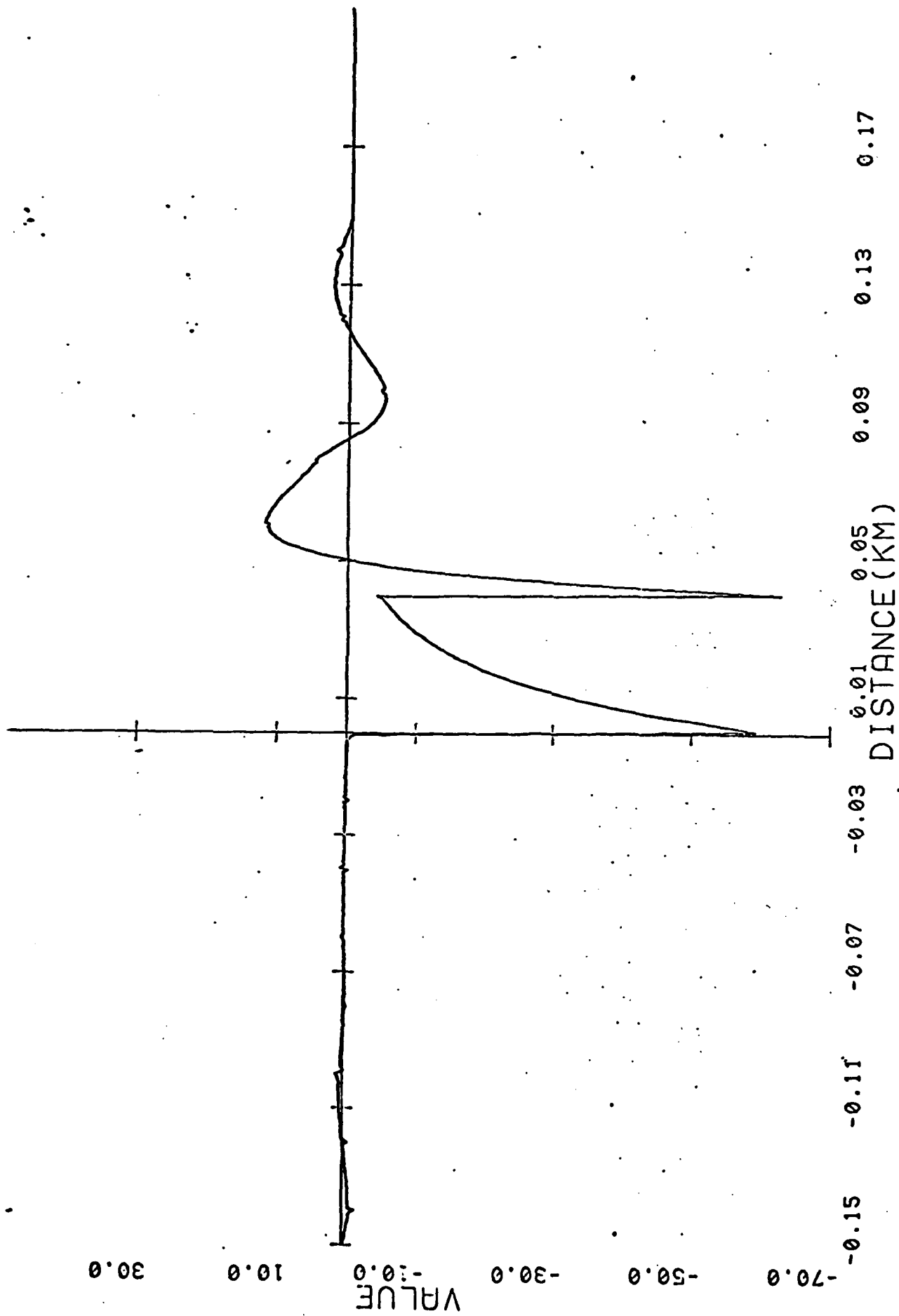




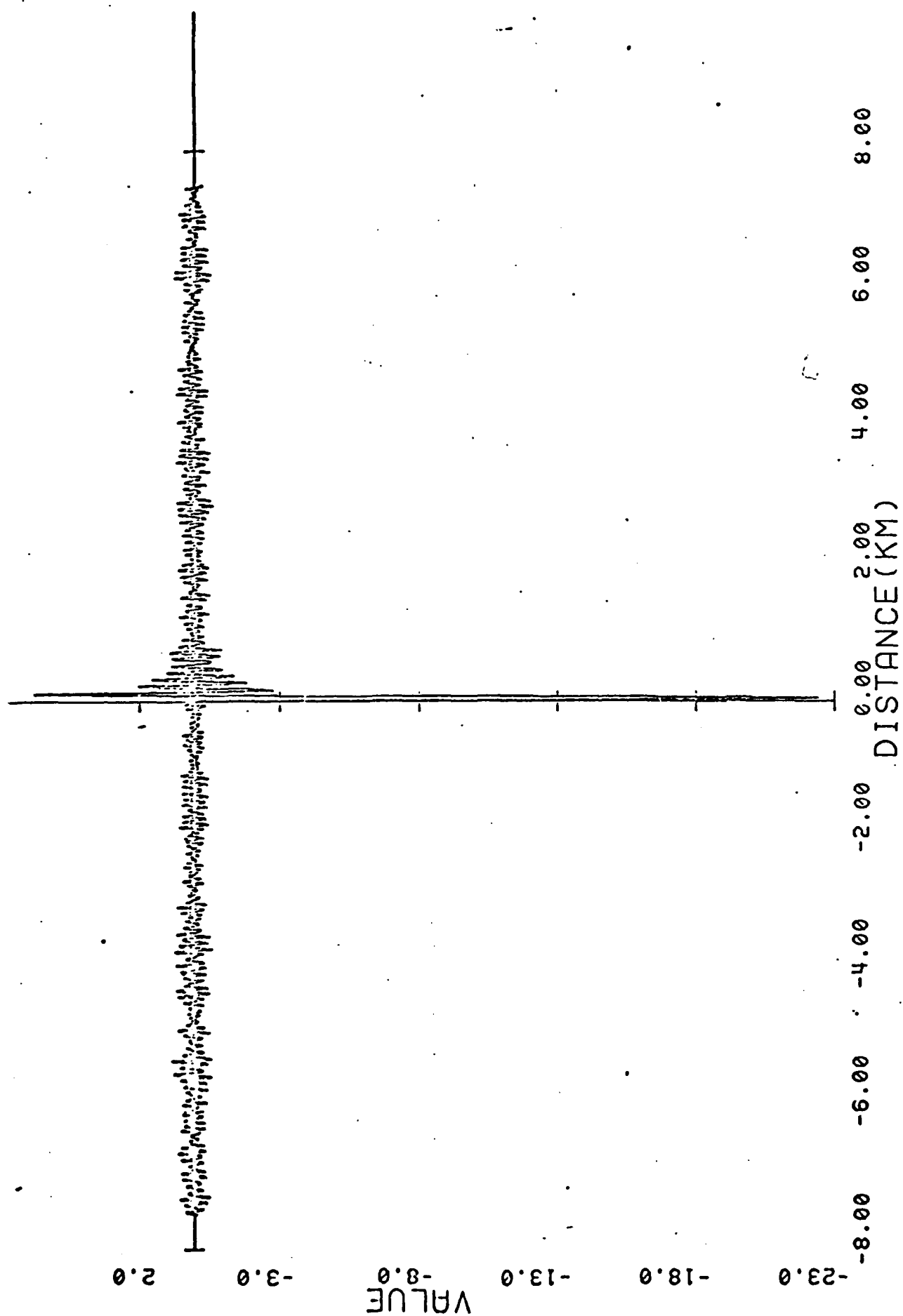
FOURIER TRANSFORM FOR REFLN FROM A DELTA FUNCTION PEN FREQ=3MHZ



FOURIER TRANSFORM FOR REFLN FROM A DELTA FUNCTION PEN FREQ=3MHZ

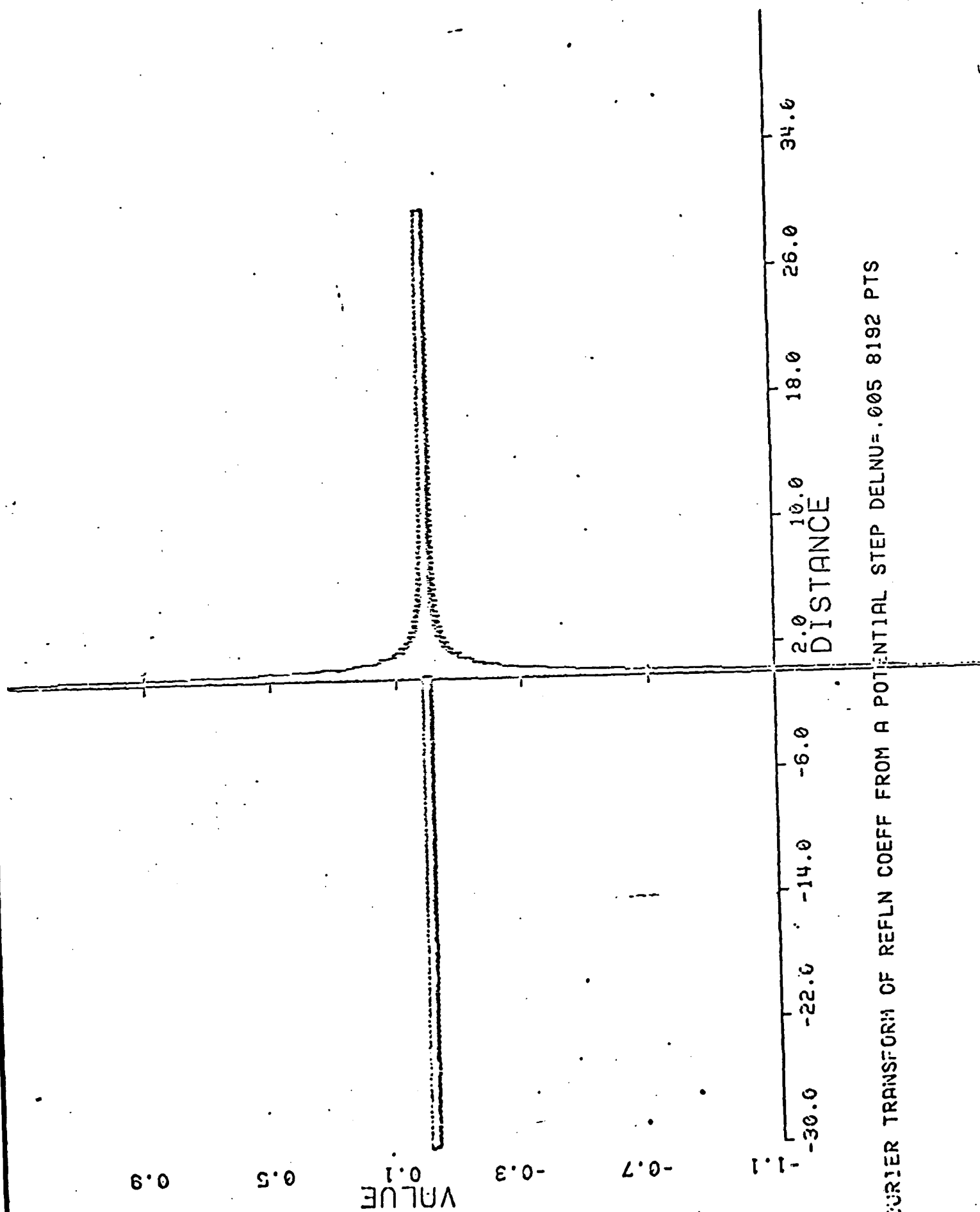


FOURIER TRANSFORM FOR REFLN FROM 2 DELTA FNS PEN FREQ=3MHZ,WIDTH=.02 KM

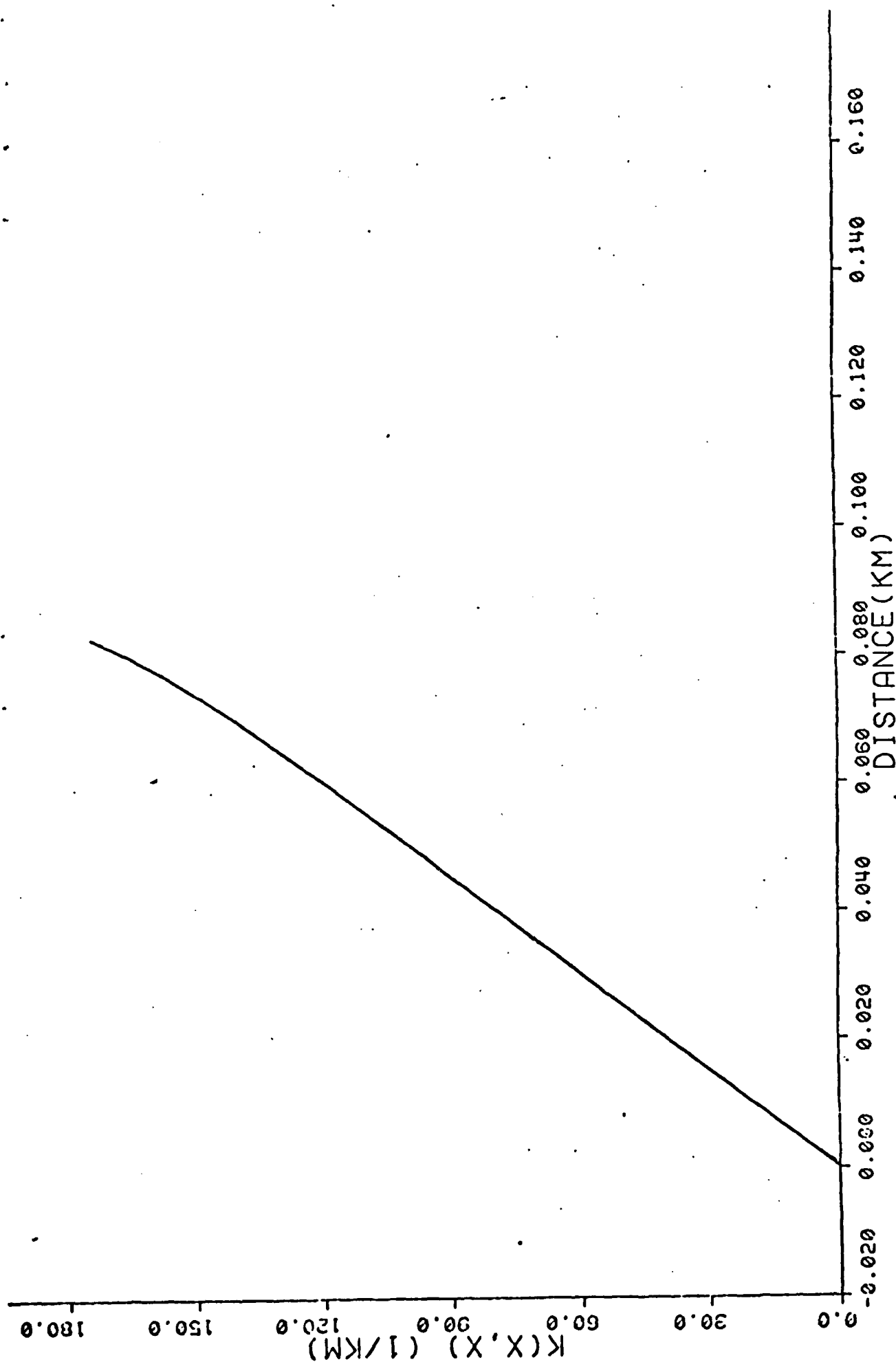


FOURIER TRANSFORM FOR REFLN FROM RECTANGULAR BARRIER PEN FREQ=3MHZ WIDTH=50KM

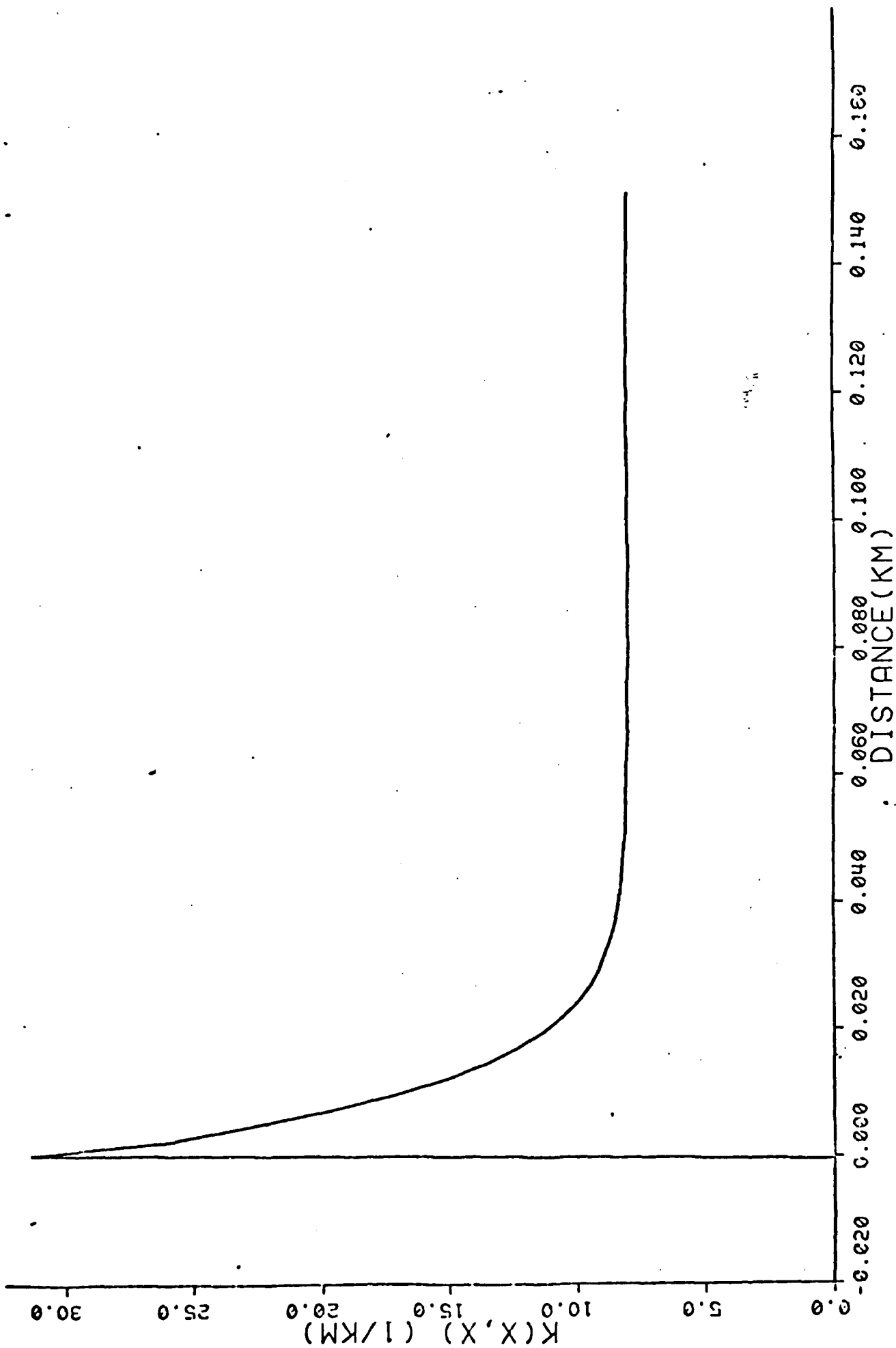
A-9



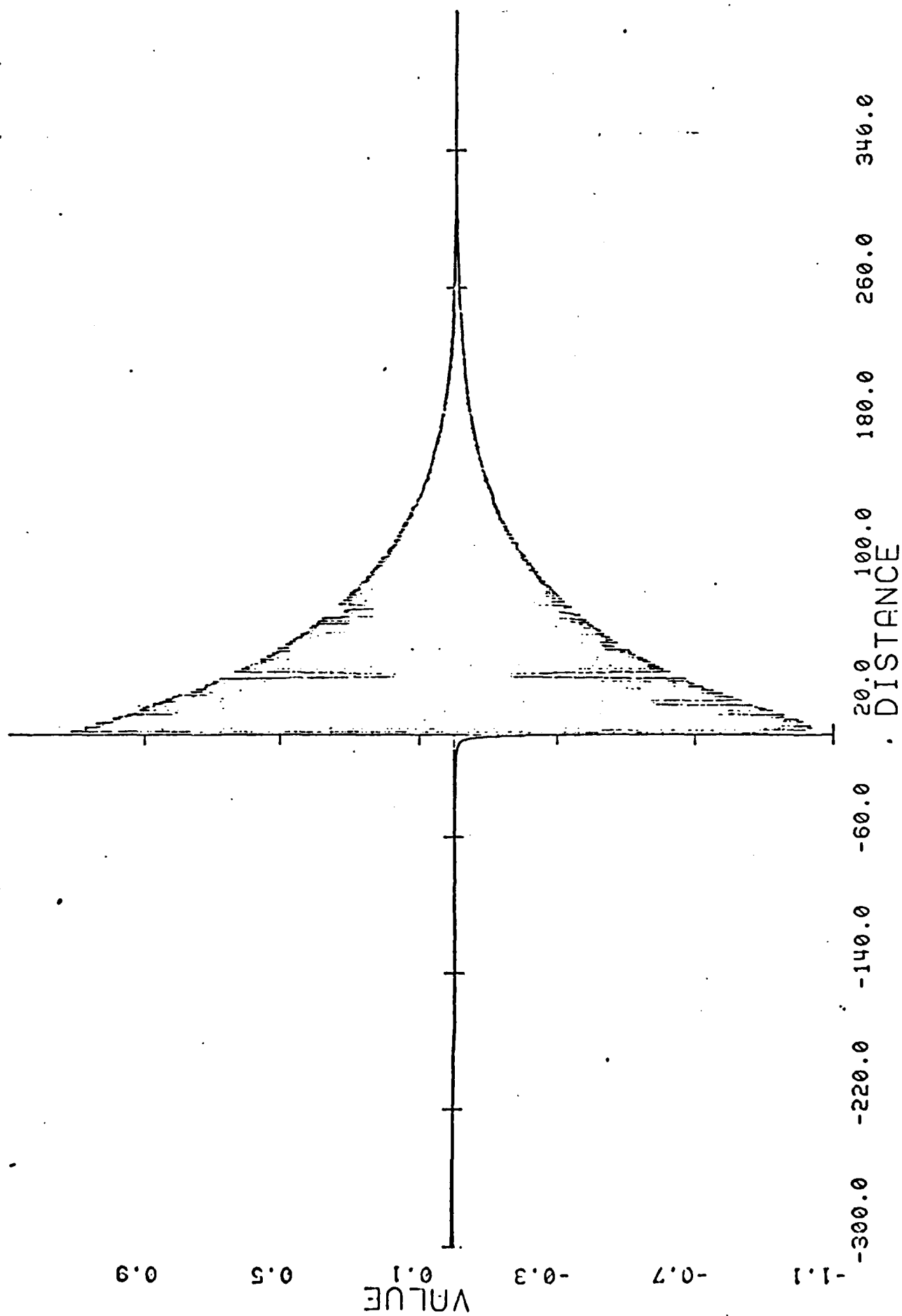
CURIER TRANSFORM OF REFLN COEFF FROM A POTENTIAL STEP DELNU=.005 8192 PTS



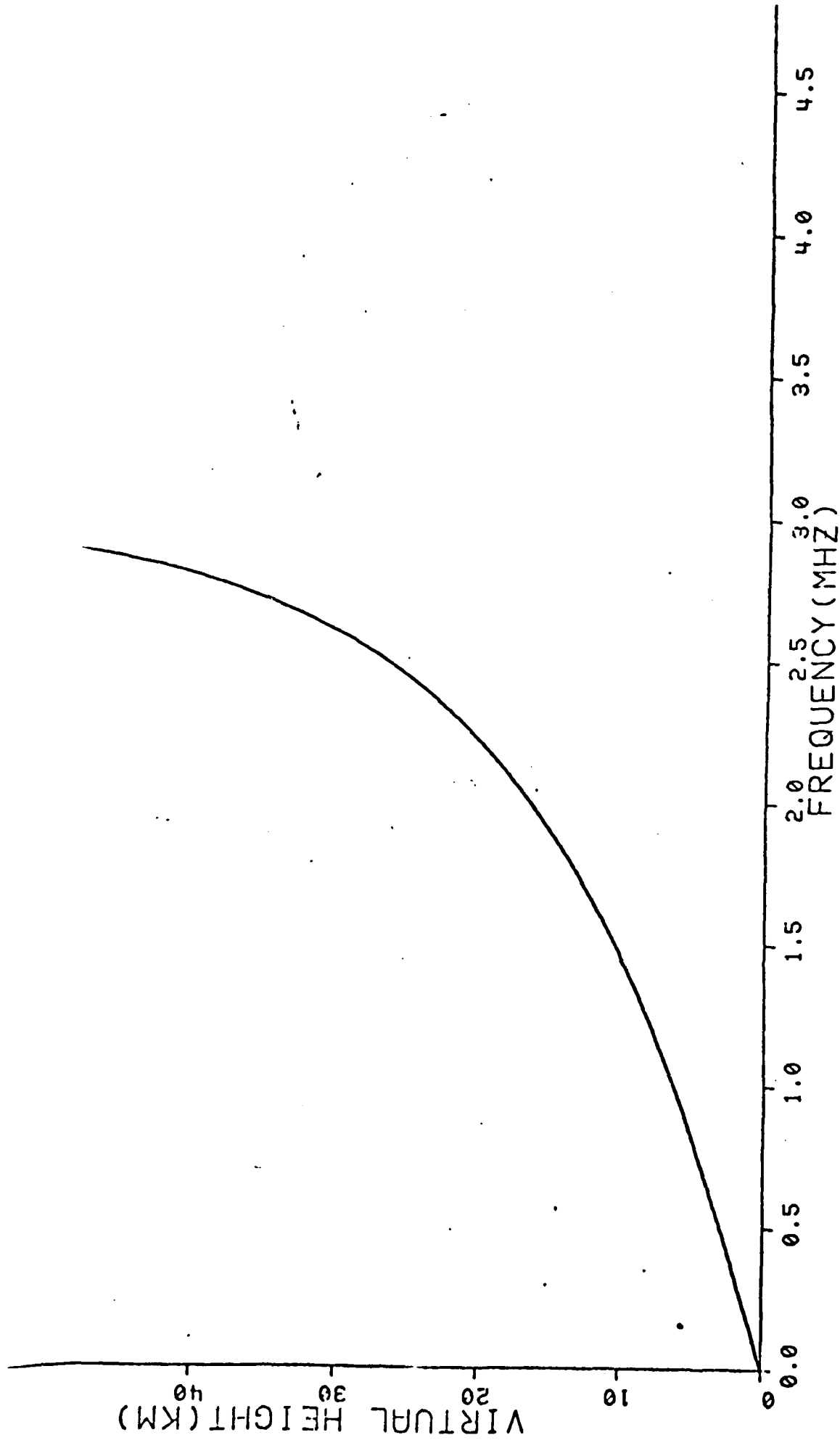
$K(X, X)$ VS X FOR SCATTERING FROM A POTENTIAL STEP OF HEIGHT 3MHZ



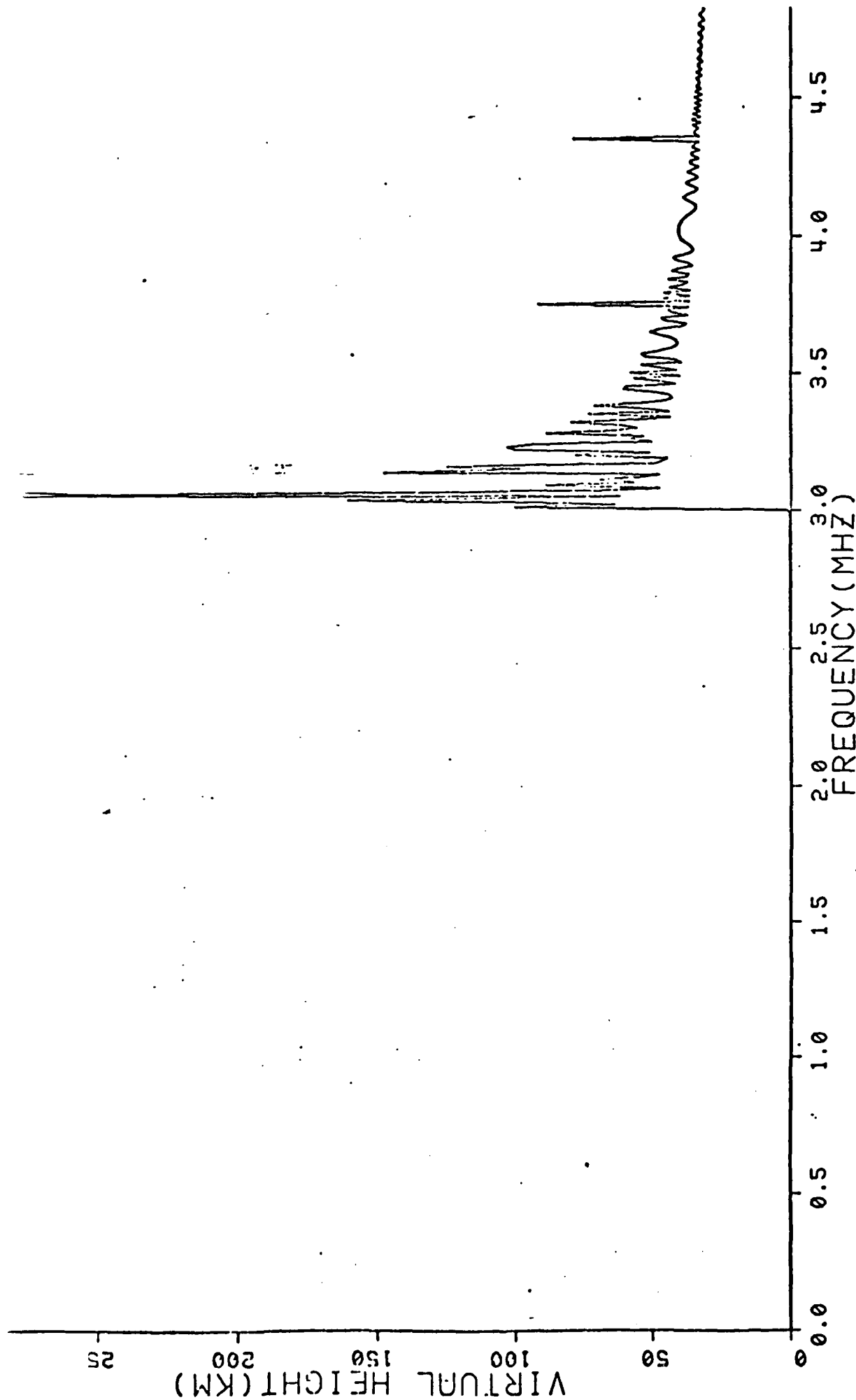
$K(X, X)$ VS X FOR REFLN FROM A 'DELTA' FUNCTION REFLN COEFF RESCALED TO $B(0)=1/2$



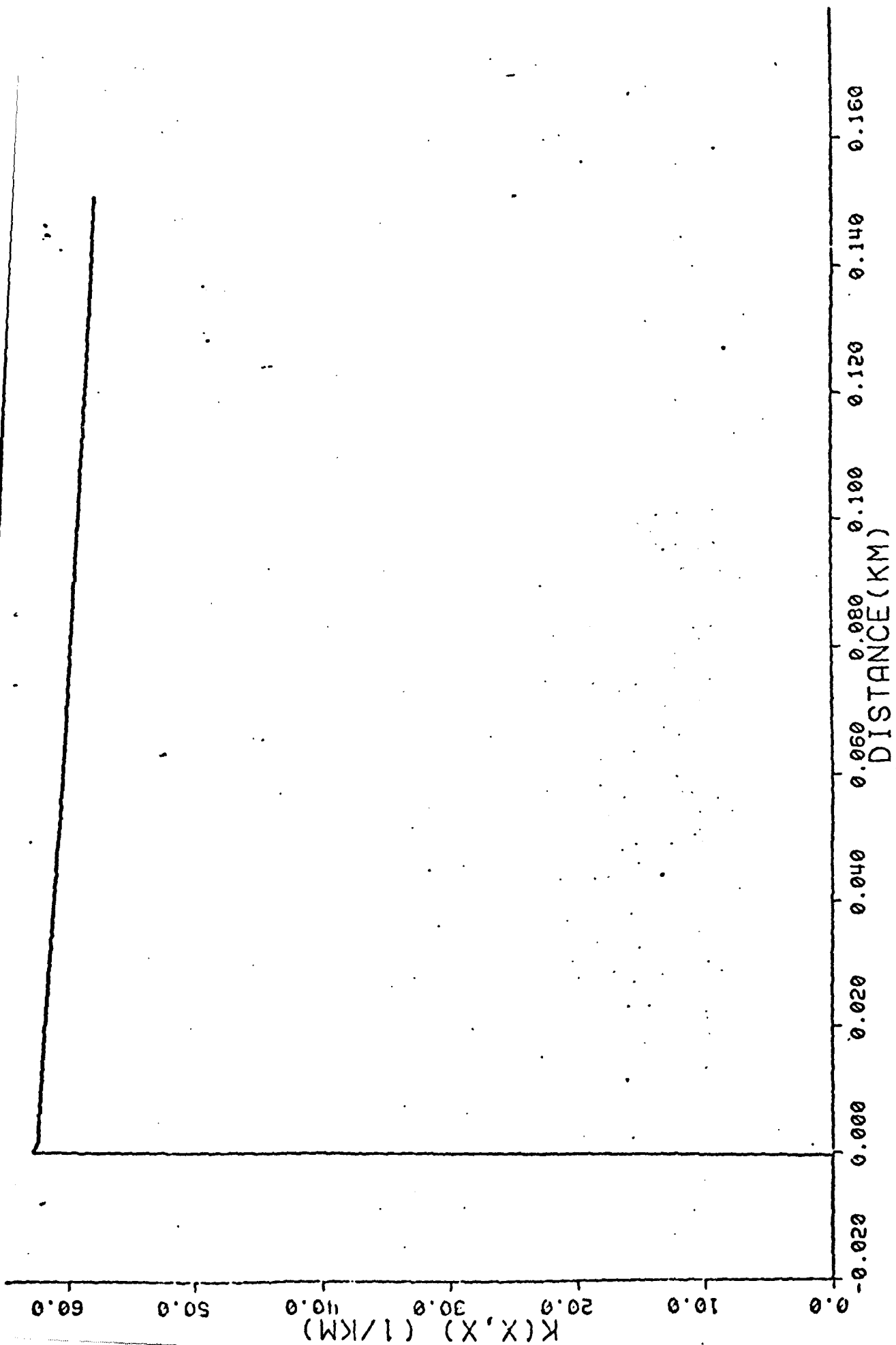
FOURIER TRANSFORM OF REFLECTION COEFFICIENT FROM A PARABOLIC POTENTIAL



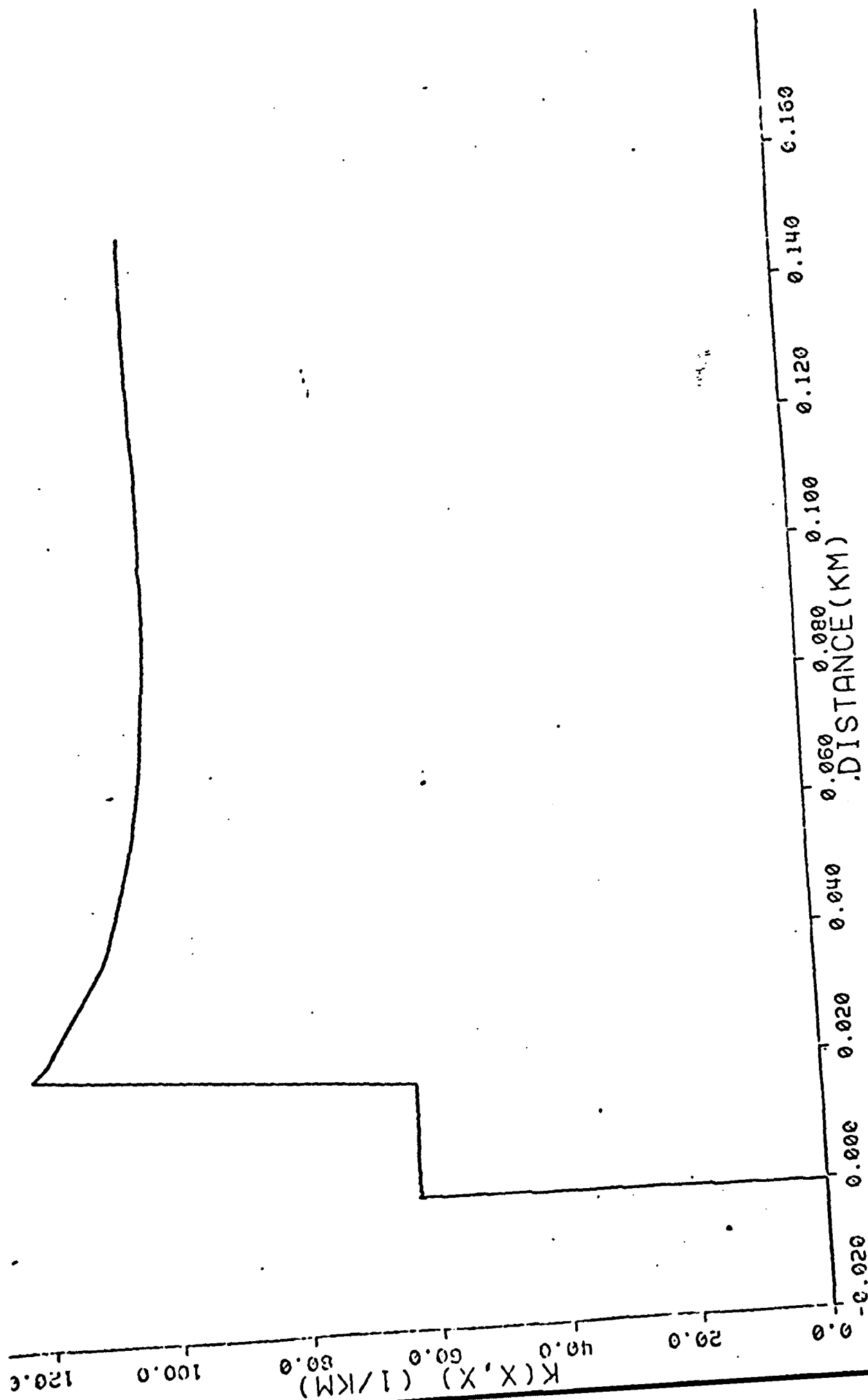
VIRTUAL HEIGHT VS FREQUENCY FOR A PARABOLIC DISTRIBUTION OF ELECTRON DENSITY
penetration frequency = 3MHZ width = 50KM



VIRTUAL HEIGHT VS FREQUENCY FOR REFLECTION FROM A RECTANGULAR DISTRIBUTION OF ELECTRON DENSITY. penetration frequency = 3MHZ, width = 50KM



K(X,X) VS X FOR SINGLE DELTA FUNCTION POTENTIAL OF PEN FREQ = 3MHZ



$K(X, X)$ VS X FOR 2 DELTA FN POTENTIAL PEN FREQ=5MHZ WIDTH=.02KM